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# VARIATIONAL PROBLEMS OF RADIATIVE GAS DYNAMICS IN THE PRESENCE OF GAS INJECTION FROM A SURFACE

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The radiant heat flux to any part of a body moving with supersonic velocity at $M \gg 1$ can be reduced by various methods [1, 2]. In connection with this, it is interesting to study ways of reducing heat flow to the frontal part of a body. One effective method here is choosing the form of the body and its flight path so as to minimize its radiant heating. Several studies (see the survey [1]) have examined problems concerning optimization of the form of a body in the presence of radiative heat transfer (without injection of gas from the surface), given different additional restrictions.

The studies [2-4] obtained relations for radiant flux to a body with allowance for the effect of a screening layer of injected gas during the disintegration of a thermally protective coating. These relations were obtained on the basis of an asymptotic solution of the equations of radiative gas dynamics. The same relations will be used here to formulate variational problems of gas dynamics in the presence of injection of gas from a surface.

Analysis of the problem shows that it is presently efficient to solve variational problems and perform comparative analyses by using an approach in which the first step involves employing approximate expressions for the radiative heat-transfer coefficients and pressure for the body that are found on the basis of analytic and numerical solutions of the equations of radiative gas dynamics. After the solution of the corresponding variational problem in the second step, the gas dynamic parameters and aerodynamic characteristics can be calculated more accurately on the basis of established numerical methods of solution with allowance for the spectral properties of the gas.

The thus-obtained preliminary results point the way to practicable methods for solving problems involving a reduction in the thermal loads on aircraft by efficiently selecting their aerodynamic shapes and the distribution of the gas injection.

Correlations to Calculate Radiant Fluxes to the Body. Using the approximation of a locally uniform plane layer when calculating radiative heat transfer in a shock layer and assuming the surface of the body to be diffusely reflecting, we have the following for the radiant flux to the surface of the body [2]:

$$q_R(t) = \pi \int_0^\infty d\nu \varepsilon_\nu \left[ 2 \int_0^{\tau_{\nu_s}} B_\nu E_2(\tau'_\nu)\, d\tau'_\nu - B_\nu(T_w) \right], \tag{1}$$
$$\tau_{\nu_c} = \int_0^{z_c} k'_\nu dz', \quad \tau_{\nu_s} = \int_0^{z_s} k'_\nu dz',$$

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where $\varepsilon_\nu$ is the spectral emissivity of the surface; $B_\nu$ is the spectral intensity of Planck radiation; $z_c$ and $z_s$ are the geometric dimensions of the injection layer and the shock layer as a whole; $E_2(t)$ is a second-order integroexponential function; $T_w$ is the temperature of the body; t and z are the coordinates along the surface of the body and along a normal to it; $\tau_\nu = \int_0^z k'_\nu dz'$ is the optical coordinate; $k_\nu'$ is the spectral absorption coefficient, with a correction for forced emission; $\tau_{\nu_c}$ and $\tau_{\nu_s}$ are the optical thicknesses of the layer of injected gas and the entire shock layer. For conditions of hypersonic entry into the atmosphere, $B_\nu(T_w)$ is small compared to the first term in (1). It is evident from (1) that the radiant flux to the body can be reduced by using coatings that have low values of $\varepsilon_\nu$. We then use the generally accepted propositions $\varepsilon_\nu = 1$, $B_\nu(T_w) \to 0$ and the approximation of the integral exponent $E_2(\tau_\nu) \simeq \exp(-2\tau_\nu)$. Considering that, under strong injection conditions [4], the optical thickness of the injected vapors $\tau_{\nu_c}$ is great in the ultraviolet part of the spectrum $\nu > \nu_1$ and that the absorption coefficient $k_{\nu_c}$ in this frequency range depends slightly on frequency and $\tau_{\nu_c} \approx \tau_c(t)$, we find that

$$q_R(t) = 2\pi \int_0^{\nu_1} d\nu q_{\nu_c}(t) \exp(-2\tau_c(t)). \tag{2}$$

The attenuation factor for the injection layer

$$K = \frac{q_R(t)}{q_c(t)} = \exp(-2\tau_c(t)) \quad \left(q_c(t) = 2\pi \int_0^{\nu_1} q_{\nu_c}(t)\, d\nu\right).$$

The case when K(t) = const was examined in [3, 5]. Here we study variational problems, when the function K(t) changes due to natural or forced injection of a substance from the surface. To calculate the optical thickness of the injected gas $\tau_c(t) = \int_0^{z_c} k'_\nu dz'$ it is necessary to have the solution of the equations of radiative gas dynamics in the injection layer. The numerical solutions in [6, 7] showed that radiative transfer in the injection layer affects the temperature profile and other parameters within a fairly narrow region around the contact surface – where, generally speaking, it is necessary to account for the influence of viscous effects as well. Thus, for our purposes, we will use the asymptotic solution [8, 9] in an injection layer obtained without the effect of radiative transfer. In von Mises variables, the required expressions take the form

$$\begin{gathered}
u(x, \psi) = [h_w(x) - h(x, \psi)]^{1/2}, \\
\frac{h(x, \psi)}{h_w} = \left[\frac{p(x)}{p(t)}\right]^{(\gamma-1)/\gamma}, \quad p = \rho T, \quad h = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}, \\
z_c(x) = \frac{1}{\sqrt{h_w}\, r(x)\, p(x)} \int_0^x \frac{\rho_w v_w(t)\, r(t) \left[\frac{p(x)}{p(t)}\right]^{(\gamma-1)/\gamma}}{\sqrt{1 - \left[\frac{p(x)}{p(t)}\right]^{(\gamma-1)/\gamma}}}\, dt,
\end{gathered} \tag{3}$$

Here, the optical thickness

$$\tau_c = k_c z_c \sqrt{\delta}\, l. \tag{4}$$

In these expressions, $\ell x$, $\sqrt{\delta \ell y}$ are coordinates directed along the surface of the body and along a normal to it; $uV_\infty\sqrt{\rho_\infty/\rho_w}$ and $vv_{w_0}$ are the components of velocity in the direction of these coordinates; $\rho\rho_{w_0}$ is the density of the gas; $\rho_\infty V_\infty^2 p$ is pressure; $h\rho_\infty V_\infty^2/\rho_{w_0}$ is enthalpy; $\delta = \rho_{w_0} v_{w_0}{}^2/\rho_\infty V_\infty{}^2$ is the injection parameter; $\ell$ is the characteristic dimension of the body; $TR_A/\ell$ is temperature; $R_A$ is the universal gas constant; $\gamma$ is the ratio of the heat capacities of the injected gas; $r\ell$ is the coordinate of the surface of the body from the symmetry axis; the subscript w denotes quantities on the surface of the body; $\infty$ denotes quantities in the incoming flow; 0 denotes quantities at the critical point.

Integrating the flux (2) over the lateral surface of the body, we find the total radiant flux

$$Q_R = 2\pi \int_0^l q_R y \frac{dx}{\cos\alpha} = 2\pi \int_0^L q_R y\, dt, \quad \operatorname{tg}\alpha = y'_x, \tag{5}$$

where x and y are rectangular coordinates connected with the critical point of the body; y = y(x) is the equation of the generatrix of the body; t is the coordinate along its surface; L is the length along its generatrix; $\ell$ is the length along the x axis; R is the radius of the middle of the body. With allowance for the assumptions made and with the use of $q_R(t)$ from [2], Eq. (5) takes the form

$$Q_R = \frac{\pi R^2 C_{H_0}(\Gamma)}{2I_0} \frac{\rho_\infty V_\infty^3}{2} I_R,$$

$$I_0 = \frac{B_0}{2(n+4)} \int_0^1 \left[1 + \frac{B_0(1-t)}{t}\right]^{-j} dt, \quad j = (n+5)/(n+4).$$

The resulting dimensionless functional

$$I_R = \left(\frac{\tau}{2}\right)^2 \int_0^1 \frac{\eta\dot{\eta}_\xi^3 d\xi}{1 + \frac{\tau^2}{4}\dot{\eta}_\xi^2} (1 - W)\exp(-2\tau_c(\xi)), \quad \dot{\eta}_\xi = d\eta/d\xi, \; B = B_0 l/r_0, \tag{6}$$

$$W = \left\{B\left(\int_\xi^1 \sqrt{1 + \frac{\tau^2}{4}\dot{\eta}_\xi^2}\, d\xi\right)\sqrt{1 + \frac{\tau^2}{4}\dot{\eta}_\xi^2}\left(\frac{\tau^2}{4}\right)^{(n+4)} \times \right.$$
$$\left. \times \left(\dot{\eta}_\xi^2 \Big/ \left(1 + \frac{\tau^2}{4}\dot{\eta}_\xi^2\right)\right)^{(n+4)} + 1\right\}^{1/(n+4)}.$$

Here, $\tau = 2R/\ell$ is the relative thickness of the body; $\eta = y/R$ and $\xi = x/\ell$ are dimensionless coordinates; n is the approximation constant in the Planck absorption coefficient; $B_0 = \Gamma(n + 4)$.

Formulation of Variational Problems. It is evident from the correlative relations for radiant flux (6) that the screening properties of the injected layer are characterized by the optical thickness $\tau_c(\xi)$, which depends on the pressure distribution on the surface of the body, the nature of the gas, the form of the body, and the injection law $G(x) = \rho_w v_w(x)$. The relations which have been found make it possible to formulate several new variational problems in order to determine the role of the injected gas in the reduction of radiative heat transfer: 1) find functions $\eta(\xi)$ and $G(\xi)$ for which the functional $I_R$ from (6) takes the lowest value and which satisfy the specified boundary conditions $\eta(0) = 0$, $\eta(1) = 1$, and (as an example) the given total discharge of the injected gas

$$2\pi \int_0^1 G(\xi)\eta(\xi)\sqrt{1 + \dot{\eta}_\xi^2}\, d\xi = R_1;$$

2) with a prescribed injection law $G(\xi)$ and assigned boundary and isoperimetric conditions, find the optimum form of the body $\eta(\xi)$ which will ensure a minimum value of $I_R$; 3) find the optimum injection law $G(\xi)$ which will ensure the maximum value of optical thickness $\tau_c(\ell)$ with a given length of the body $\ell$, a given shape $\eta = \eta(\xi)$, and a given discharge of the injected gas $R_1$ = const.

Variational problems with other restrictions can be studied within the framework of these formulations, including problems with restrictions on the volume of the body, the ballistic factor, the area of the wetted surface, the momentum or energy of the injected gas, etc. To illustrate, we will stop to discuss several examples.

Solution of a Problem of Minimizing Radiant Flux for a Power-Law Thin Body. We will examine the determination of the optimum injection law that will ensure the shape of the "ef-

TABLE 1

| $n$ | $\lambda$ | $R, l$ | | | $V, l$ | | | $S, l$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $\alpha_{opt}$ | $\beta$ | $\mu$ | $\alpha_{opt}$ | $\beta$ | $\mu$ | $\alpha_{opt}$ | $\beta$ | $\mu$ |
| 0 | 3,33 | 0,979 | 0,984 | 0,055 | 0,927 | 0,945 | 0,183 | 0,911 | 0,933 | 0,223 |
| 4 | 6,00 | 0,984 | 0,996 | 0,024 | 0,957 | 0,989 | 0,065 | 0,948 | 0,987 | 0,078 |
| 8 | 8,67 | 0,988 | 0,998 | 0,016 | 0,969 | 0,995 | 0,040 | 0,963 | 0,994 | 0,048 |

fective" body formed as a result of injection which corresponds to the minimum total radiant flux to the lateral surface. The "effective" body formed as a result of injection will be assumed to be thin, i.e., $\tau \ll 1$. Then we can use the limiting expression obtained in [10] to determine the total radiant flux to the surface of the "effective" body (surface of contact discontinuity). From (6)

$$Q_c = \Pi_\infty bk\pi R^3 \left(\frac{\tau}{2}\right)^{(m-2)} I, \quad I = \int_0^1 (1-\xi)\,\eta\dot{\eta}_\xi^m\, d\xi,$$

$$m = 2(n+4)+3,\; k = (n+4)^{-1},\; \eta = y/R,\; \xi = x/l.$$

Here, x and y are the coordinates along the direction of motion of the incoming gas flow and perpendicular to this direction; $\ell$ is the length of the "effective" body; R is the radius of its middle. The remaining quantities were determined in [10], where the authors optimized the aerodynamic shapes in the class of thin power-law bodies with different isoperimetric conditions. These results are fully valid in our case for the "effective" body. The form of the actual body is calculated with allowance for the data in [10], the thickness of the injection layer in accordance with Eq. (3), and the energy balance

$$q_c(\xi)\exp(-2\tau_c(\xi)) = G_w(\xi) H_{ef}. \tag{7}$$

For power laws of injection and power-law bodies

$$y(x) = kx^\beta, \quad 1/2 \leqslant \beta \leqslant 1, \quad G_w(x) = ax^\mu, \quad y_x' = k\beta x^{(\beta-1)}, \quad p(x) = y_x'^2$$

the expression for $z_c$ is simplified. With the substitutions $t/x = v$, $\alpha = (1-\beta)(2(\gamma-1)/\gamma)$, by calculating the integrals we obtain

$$z_c = \frac{2a}{\sqrt{h_w}\, k^2\beta^2\sqrt{\alpha}}\, x^{\mu+3-2\beta}\left\{1 - \frac{\beta+\mu+\alpha}{3} + \ldots\right\}.$$

As a first approximation, condition (7) of energy balance on the surface of the body gives $q_c(\xi) = G(\xi)H_{ef}$. Using the relation $q_c(x) \approx (y_x')^\lambda$ from [2], we find the relation between the exponents $\lambda(\beta-1) = \mu$, $\lambda = (2/3)(n+5)$, $\mu + 3 - 2\beta = \alpha_{opt}$.

Table 1 shows values of $\mu$ and $\beta$ corresponding to different $\lambda$ and $\alpha_{opt}$ for different isoperimetric conditions when the radius and length of the body, the volume and length of the body, and the area of the wetted surface and the length are given.

It follows from the calculations that an increase in $\lambda$ is accompanied by a decrease in the absolute value of the exponent $\mu$ characterizing the distribution of injection. It should be noted that the form of the body is close to conical. The body is most blunt with assigned S and $\ell$, while it is sharpest with assigned R and $\ell$. At the nose of the body, $G_w \to \infty$. This is connected with violation of the proposition that the body is thin, but it does not interfere with calculation of the aerodynamic characteristics of the body.

Solution of the Problem of the Maximum Optical Thickness of the Injection Layer. We will examine the following problem. Find optimum functions $\eta(\xi)$ and $G(\xi)$ which will ensure the maximum optical thickness $\tau_c(\ell)$ with a given discharge of the injected gas $R_1$ = const.

Using (3) and (4) and the necessary conditions for the existence of the extremum, we find that the solution of the Euler equation is a cone. Here, due to the linear dependence of $\tau_c(\ell)$ on $G(\xi)$, the form of the latter is not found from the Euler equation. If we assume that $G(\xi)$ = const, then its value can be determined from the specified discharge, while the solution of the Euler equation is found in parametric form. The solution of this problem is found for power laws of injection and power-law thin bodies. The functional $\tau_c(\ell)$ has the form

$$\tau_c(l) = B\frac{al^{\mu-2\beta+3}}{k^2\beta^2}\int\limits_0^1 \frac{v^{\mu+\beta}v^{\frac{\gamma-1}{\gamma}(1-\beta)}\,dv}{\sqrt{1-v^{\frac{2(\gamma-1)}{\gamma}(\beta-1)}}} \quad (v = l/\xi,\ B = k_c\sqrt{\delta l}).$$

Using expansions into a series, we obtain the approximate expression

$$\tau_c(l) = \frac{Bal^{\mu+3-2\beta}}{\beta^2k^2}\frac{2}{\sqrt{\alpha}}\left(1-\frac{\mu+\beta+\alpha}{3}\right) \quad (\alpha = (2(\gamma-1)/\gamma)(1-\beta)). \tag{8}$$

The total rate of flow of the substance across the lateral surface

$$2\pi kal^{\mu+\beta+1}/(\mu+\beta+1) = R_1. \tag{9}$$

With a specified value of $\beta$, the condition of the maximum of $\tau_c(\ell)$ from (8) and (9) is reached at $\mu_{opt} = (1-\beta)/\gamma$. The sufficient condition for the existence of the extremum $\partial^2\tau_c/\partial\mu^2 = -2/3$ confirms that this value of $\mu$ ensures the maximum of the function $\tau_c(\ell)$. If $\mu$ and $\beta$ are arbitrary, then $\tau_c(\ell)$ has a maximum at $\mu = (1-\beta)/\gamma$, while $\beta$ is determined from the necessary condition of the existence of the extremum $\tau_c(\ell)$: $\beta/\sqrt{1-\beta}(4-3\beta-3\beta^2-\beta^3) = 0$. This equation has the roots $\beta_1 = 0$ and $\beta_{2,3} = 5^{1/3}-1$. The first root obviously has no physical meaning, while the second root $\beta_2 = 0.73$ is the optimum value of the injection index $\mu_{opt} \approx 0.2$.

The simplest solutions were obtained above. A more complete solution of the above-stated variational problems can be found by well-known numerical methods [11-13].

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